JOURNAL OF COMPUTATIONAL AND APPLIED MATHEMATICS

ELSEVIER Journal of Computational and Applied Mathematics 118 (2000) 169–173

Hypergeometric functions and the trinomial equation

M.L. Glasser

Physics Department, Clarkson University, Potsdam, NY 13699-5280, USA

Received 10 February 1999; received in revised form 17 May 1999

Abstract

It is shown that the roots of the trinomial equation

$$x^n - x + t = 0$$

are finite sums of generalized hypergeometric functions for each positive integer n. As a consequence, we demonstrate a class of algebraic identities satisfied by certain of these functions.

Keywords: Algebraic equation; Hypergeometric functions

1. Introduction

The aim of this note is to express the roots of any trinomial equation

$$x^n - x + t = 0$$

as a finite sum of generalized hypergeometric functions. In many cases, and several explicit examples will be given below, one of the roots can be expressed as a single hypergeometric function. By means of a Tchirnhausen transformation, this provides relatively simple expressions for the roots of the general equation of degree less than six. On the other hand, we obtain in this way a number of (in principle, infinitely many) algebraic identities satisfied by certain hypergeometric functions. The quintic equation, in particular, has been studied extensively for at least two hundred years and a number of transcendental procedures are known for solving it (Hermite's, described, e.g., in [2] is probably the most familiar one). For this case, we recover a century old formula obtained by the method of differential invariants.

PII: S0377-0427(00)00287-9

2. The formula

Without loss of generality, up to degree 6, it is sufficient to find at least one root to the reduced equation

$$x^N - x + t = 0 \quad (N = 2, 3, 4, \ldots). \tag{1}$$

Letting $x = \zeta^{-1/(N-1)}$, we easily find that (1) becomes

$$\zeta = e^{2\pi i} + t\phi(\zeta), \tag{2}$$

where

$$\phi(\zeta) = \zeta^{N/(N-1)}. \tag{3}$$

Lagrange's theorem states that for any function f analytic in a neighborhood of a root of (2)

$$f(\zeta) = f(e^{2\pi i}) + \sum_{n=1}^{\infty} \frac{t^n}{n!} \frac{d^{n-1}}{da^{n-1}} [f'(a)[\phi(a)]^n]_{a=e^{2\pi i}}. \tag{4}$$

We now simply let $f(\zeta) = \zeta^{-1/(N-1)}$, carry out the elementary differentiations (noting that $D_k x^p = \Gamma(p+1)x^{p-k}/\Gamma(p-k+1)$) and we come up with the root

$$x_1 = \exp[-2\pi i/(N-1)] - \frac{t}{N-1} \sum_{n=0}^{\infty} \frac{(te^{2\pi i/(N-1)})^n}{\Gamma(n+2)} \frac{\Gamma(Nn/(N-1)+1)}{\Gamma(n/(N-1)+1)}. \tag{5}$$

($N-2$ further roots are found by replacing $\exp(2\pi i/(N-1))$ by the other $(N-1)$st roots of unity, and the remaining root from the relation $\sum x_j = \delta_{N,2}$.) By the use of Gauss' multiplication theorem, the infinite series can be broken up into a (finite) sum of hypergeometric functions.

$$x_1 = \omega^{-1} - \frac{t}{(N-1)^2} \sqrt{\frac{N}{2\pi(N-1)}} \sum_{q=0}^{N-2} \left(\frac{\omega t}{N-1}\right)^q N^{qN/(N-1)}$$

$$\times \frac{\prod_{k=0}^{N-1} \Gamma((Nq/(N-1)+1+k)/N)}{\Gamma(q/(N-1)+1) \prod_{k=0}^{N-2} \Gamma((q+k+2)/(N-1))}$$

$$\times {}_{N+1}F_N \left[\frac{qN/(N-1)+1}{N}, \ldots, \frac{qN/(N-1)+N}{N}, 1;\right.$$

$$\left.\frac{q+2}{N-1}, \ldots, \frac{q+N}{N-1}, \frac{q}{N-1}+1; \left(\frac{t\omega}{N-1}\right)^{N-1} N^N\right],$$

where $\omega = \exp(2\pi i/(N-1))$. In practice, ${}_{N+1}F_N$ will always be reducible to at least ${}_NF_{N-1}$. Hence, the root is a sum of at most $N-1$ hypergeometric functions. The one technical point is that the convergence of these series requires that t be "sufficiently small", but this can be overcome by certain hypergeometric identities tantamount to analytic continuation.

The procedure leading to (5) is well known. Indeed, in [3] it is shown that for any β and w, the root of

$$1 - x + wx^{\beta} = 0$$

that reduces to 1 when $w \to 0$ is

$$x = 1 + \sum_{n=1}^{\infty} \binom{\beta n}{n-1} \frac{w^n}{n}.$$

The main contribution of this note is to sum this series in terms of hypergeometric functions when $\beta = N$.

3. Examples

3.1. $N = 2$

$$x^2 - x + t = 0.$$

Here we have

$$x_1 = 1 - t \sum_{n=0}^{\infty} \frac{t^n}{\Gamma(n+1)} \frac{\Gamma(2n+1)}{\Gamma(n+1)}. \tag{6}$$

However, by Gauss' formula

$$\Gamma(2n+1) = 4^n (1/2)_n (1)_n \quad ((n)_k = \Gamma(n+k)/\Gamma(n)), \tag{7}$$

so

$$x_1 = 1 - t\ {}_2F_1(\tfrac{1}{2}, 1; 2; 4t). \tag{8}$$

Since

$${}_2F_1(\tfrac{1}{2}, 1; 2; z) = \frac{2}{z} \begin{cases} 1 - \sqrt{1-z} & |z| \leqslant 1, \\ 1 - \mathrm{i}\sqrt{z-1} & |z| > 1, \end{cases} \tag{8}$$

we reproduce the quadratic formula. Note that the second root comes from $x_1 + x_2 = 1$.

3.2. $N = 3$

$$x^3 - x + t = 0.$$

By separating the sum in (5) into sums over the even and odd values of n we obtain

$$x_1 = -1 + \frac{t}{2} \sum_{n=0}^{\infty} \frac{\Gamma(3n+1) t^{2n}}{\Gamma(n+1)\Gamma(2n+2)} + \frac{t^2}{2} \sum_{n=0}^{\infty} \frac{\Gamma(3n+5/2) t^{2n}}{\Gamma(n+3/2)\Gamma(2n+3)}. \tag{9}$$

By breaking up the gamma functions of multiple argument by using Gauss' multiplication theorem, the sums are easily identified as hypergeometric series:

$$x_1 = -1 - \frac{t}{2}\ {}_2F_1(\tfrac{1}{3}, \tfrac{2}{3}; \tfrac{3}{2}; 27t^2/4) + \frac{3t^2}{8}\ {}_3F_2(\tfrac{5}{6}, \tfrac{7}{6}, 1; \tfrac{3}{2}, 2; 27t^2/4). \tag{10}$$

However, from [4] we find

$$\begin{aligned} {}_2F_1(\tfrac{1}{3}, \tfrac{2}{3}; \tfrac{3}{2}; z) &= \frac{3}{\sqrt{z}} \sin\left(\frac{1}{3} \sin^{-1} \sqrt{z}\right), \\ {}_3F_2(\tfrac{5}{6}, \tfrac{7}{6}, 1; \tfrac{3}{2}, 2; z) &= \frac{18}{z} \left[\cos\left(\frac{1}{3} \sin^{-1} \sqrt{z}\right) - 1\right] \end{aligned} \tag{11}$$

and we therefore have the three roots

$$x_1 = -\frac{1}{\sqrt{3}}\sin\left[\frac{1}{3}\sin^{-1}(t\sqrt{27}/2)\right] - \cos\left[\frac{1}{3}\sin^{-1}(t\sqrt{27}/2)\right],$$
$$x_2 = -\frac{1}{\sqrt{3}}\sin\left[\frac{1}{3}\sin^{-1}(t\sqrt{27}/2)\right] + \cos\left[\frac{1}{3}\sin^{-1}(t\sqrt{27}/2)\right], \tag{12}$$
$$x_3 = \frac{2}{\sqrt{3}}\sin\left[\frac{1}{3}\sin^{-1}(t\sqrt{27}/2)\right].$$

Once again, for $t > 2/\sqrt{27}$ Eq. (10) must be analytically continued to obtain the correct form of (12). This amounts to writing $\sin^{-1} z = \pi/2 - \mathrm{i}Ln(z + \sqrt{z^2 - 1})$.

4. Conclusion

For $N = 2, 3, 4$, Eq. (5) does not appear to be preferable to the standard formulas, but for $N = 5$, e.g., we get the root

$$x = t\ {}_4F_3\left[\begin{matrix}\frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}; \\ \frac{1}{2}, \frac{3}{4}, \frac{5}{4};\end{matrix}\ \frac{3125t^4}{256}\right] \tag{13}$$

in an elementary fashion with considerably less difficulty than by following the procedure in [2]. This expression reproduces the series for the root of the quintic obtained by Cockle [1] in 1860 using the method of differential resolvents. It might also be pointed out that the above procedure carries over in a trivial way to the trinomial equation

$$y^N - ay^{N-1} + a = 0, \tag{14}$$

where $y = 1/x,\ a = 1/t$.

We conclude by pointing out that the hypergeometric function appearing in (13)

$$f(x) = {}_4F_3(\tfrac{1}{5}, \tfrac{2}{5}, \tfrac{3}{5}, \tfrac{4}{5}; \tfrac{1}{2}, \tfrac{3}{4}, \tfrac{5}{4}; x), \tag{14}$$

consequently, obeys the identity

$$f(x) = \left[\frac{c}{x}(f(x) - 1)\right]^{1/5}, \tag{15}$$

where $c = \frac{3125}{256}$. Furthermore, we have shown the existence of at least one such identity for each positive integer n.

Acknowledgements

I am grateful for the hospitality of the University of Melbourne where this work was begun, and thank Professor Hari Srivastava for pointing out Ref. [3].

References

[1] J. Cockle, Sketch of a Theory of Transcendental Roots, Philos. Mag. 20 (1860) 145.
[2] H.T. Davis, Introduction to nonlinear differential and integral equations, US Atomic Energy Commission, September 1960.
[3] G. Pólya, G. Szegö, Aufgaben and Lehrsätze Aus der Analysis, Dover Publishers, New York, 1945, Problem 211.
[4] A.P. Prudnikov et al., Integrals and Series, Vol. 3, Gordon and Breach, London, 1990.